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THE ISOPERIMETRIC QUOTIENT AND SOME CLASSICAL BANACH SPACES†

BY

CARSTEN SCHÜTT

Department of Mathematics, Oklahoma State University, Stillwater, OK 74078-0613, USA

ABSTRACT

We obtain estimates for projection bodies of unit balls of certain Banach spaces. Some of these estimates involve the isoperimetric quotient.

We consider centrally symmetric, convex bodies K in $\mathbf{R}^n$ with the origin as center. For such a convex body K we can always find a linear transform T such that $\mathrm{vol}_{n-1}(T(K) \cap H_\xi)$ is essentially the same for all $n-1$ dimensional hyperplanes H_ξ, ξ is the vector orthogonal to H_ξ [Hen]. If we consider orthogonal projections $P_\xi(K)$ of K instead of $K \cap H_\xi$ we cannot find a corresponding transform T. The most obvious example is the cube or unit ball of l_∞^n. The projection body of a convex body K is the convex body whose support hyperplane orthogonal to ξ has distance $\mathrm{vol}_{n-1}(P_\xi(K))$ from the origin. This notion was introduced by Minkowski [BF]. A paper of Bourgain and Lindenstrauss [BL] contains a survey of projection bodies. We investigate which are the projection bodies of unit balls of classical normed spaces. For l_p^n, $1 \leqq p < \infty$, we obtain that the projection body is equivalent to the Euclidean sphere, i.e. the volume of $P_\xi(B_p^n)$ is essentially independent of the direction ξ. Since the projection body of the cube or B_∞^n is a multiple of the cube there must be a drastic change when p tends toward ∞. In fact, if the Orlicz function M satisfies a Δ_2-condition then the projection body of the unit ball B_M^n of l_M^n is equivalent to the Euclidean sphere. Therefore we study variations of these spaces. In the case of l_p^n these variations are simply the Lorentz spaces $l_{p,q}^n$. We get projection bodies that are not equivalent to the Euclidean sphere. We apply

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our results to get estimates for successive quotients of quermassintegrals. We would like to thank Joram Lindenstrauss, Jerusalem, and Keith Ball, College Station, for discussions.

0. Preliminaries

In this paper we consider $\mathbf{R}^n$ equipped with a norm or normed spaces E that are naturally identified with $\mathbf{R}^n$. Therefore it is clear what we understand by the Lebesgue measure on a normed space. A basis $\{e_i\}_{i=1}^n$ is 1-unconditional if we have for all $\varepsilon_i = \pm 1$, $a_i \in \mathbf{R}$, $i = 1, \ldots, n$

$$\left\| \sum_{i=1}^n a_i e_i \right\| = \left\| \sum_{i=1}^n \varepsilon_i a_i e_i \right\|$$

and 1-symmetric if, in addition,

$$\left\| \sum_{i=1}^n a_i e_i \right\| = \left\| \sum_{i=1}^n a_{\pi(i)} e_i \right\|$$

for all permutations. We denote $\lambda(k) = \| \sum_{i=1}^k e_i \|$. $B_E(x)$ is the unit ball of E with center x. $B_{p,q}^n$ is the unit ball of the Lorentz space $l_{p,q}^n$ with norm

$$\| x \|_{p,q} = \left(\sum_{i=1}^n |x_i^*|^q i^{q/p-1} \right)^{1/q}, \qquad 1 \leqq q \leqq p < \infty$$

where x_i^*, $i = 1, \ldots, n$, is the decreasing rearrangement of $|x_i|$, $i = 1, \ldots, n$. We denote $p' = p/(p-1)$. An Orlicz function M is a convex function from $[0, \infty)$ to $\mathbf{R}^+$ with $M(t) = 0$ if and only if $t = 0$. B_M^n is the unit ball of the Orlicz space l_M^n. The norm satisfies

$$\| x \|_M = 1 \qquad \text{if and only if } \sum_{i=1}^n M(|x_i|) = 1,$$

M^* is the dual function. B_2^n is the Euclidean unit ball and σ the Haar measure on its boundary ∂B_2^n. μ is usually the surface measure on the boundary ∂K of a convex body K, the restriction of the $n-1$ dimensional Hausdorff measure to ∂K [Fe]. We use the fact that the surface area of a convex body K that is contained in a convex body C is smaller than the surface area of C. The exterior normal at a point $x \in \partial K$ is denoted by $N(x)$. $N(x)$ exists almost everywhere.

1. The isoperimetric quotient

The isoperimetric quotient iq(C) of a convex body C in $\mathbf{R}^n$ [Had, p. 269] is

$$\mathrm{iq}(C)=\frac{\mathrm{vol}_{n-1}(\partial C)}{\mathrm{vol}_n(C)^{(n-1)/n}} .$$

THEOREM 1.1. *Let $\{e_i\}_{i=1}^n$ be a basis of E, $\Gamma\subseteq\{\varepsilon \mid \varepsilon_i=\pm 1, i=1,\ldots,n\}$ and Π a subset of the set of permutations of $\{1,\ldots,n\}$ such that*

$$\left\|\sum_{i=1}^n a_i e_i\right\|=\left\|\sum_{i=1}^n \varepsilon_i a_{\pi(i)} e_i\right\| \qquad \text{for all } \varepsilon\in\Gamma,\quad \pi\in\Pi \quad\text{and}\quad a_i\in\mathbf{R}. \tag{1.1}$$

Assume that

$$\frac{1}{c_1}\|a\|_2\leqq \operatorname*{Ave}_{\varepsilon\in\Gamma}\left|\sum_{i=1}^n a_i\varepsilon_i\right|\leqq c_2\|a\|_2 \qquad \text{for all } a\in\mathbf{R}^n \tag{1.2}$$

and that for every i, k with $1\leqq i,k\leqq n$ there is a $\pi\in\Pi$ so that $\pi(i)=k$ and $\#\{\pi \mid \pi(i)=k\}$ does not depend on i and k. Then we have

$$\max_{\xi,\eta}\frac{\mathrm{vol}_{n-1}(P_\xi(B_E))}{\mathrm{vol}_{n-1}(P_\eta(B_E))}\leqq \tfrac{1}{2}c_1c_2\frac{\mathrm{iq}(B_E)}{\sqrt{n}} . \tag{1.3}$$

REMARK 1.2. For unit balls B_E of spaces E that have a 1-symmetric basis we get with an absolute constant C

$$C^{-1}\frac{\mathrm{iq}(B_E)}{\sqrt{n}}\leqq \max_{\xi,\eta}\frac{\mathrm{vol}_{n-1}(P_\xi(B_E))}{\mathrm{vol}_{n-1}(P_\eta(B_E))}\leqq C\frac{\mathrm{iq}(B_E)}{\sqrt{n}} . \tag{1.4}$$

The left-hand inequality follows from Cauchy's surface formula and a result of Hensley. Indeed, by Cauchy's surface formula we get

$$\begin{aligned}\mathrm{vol}_{n-1}(\partial C)&=\mathrm{vol}_{n-1}(B_2^{n-1})^{-1}\int_{\partial B_2^n}\mathrm{vol}_{n-1}(P_\xi(C))d\mu(\xi)\\ &\leqq\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\max_\xi\ \mathrm{vol}_{n-1}(P_\xi(C))\end{aligned}$$

and by a result of Hensley [Hen] and the symmetricity of the basis we have

$$\mathrm{vol}_n(C)^{(n-1)/n}\geqq c\,\mathrm{vol}_{n-1}(C\cap H_{e_1})=c\,\mathrm{vol}_{n-1}(P_{e_1}(C));$$

H_{e_1} denotes the hyperplane orthogonal to e_1. The two inequalities imply the left-hand inequality of Remark 1.2.

Theorem 1.1 applies in particular to the following class of spaces. We say that a finite-dimensional Banach space is a symmetric matrix space if there is a basis $\{e_{ij}\}_{i,j=1}^{n,m}$ such that we have for all signs ε, θ, all permutations π, σ, and all $a_{ij} \in \mathbf{R}$

$$\left\| \sum_{i,j=1}^{n,m} a_{ij} e_{ij} \right\| = \left\| \sum_{i,j=1}^{n,m} \varepsilon_i \theta_j a_{\pi(i)\sigma(j)} e_{ij} \right\| .$$

Examples are the Schatten classes C_p^n and tensor products of 1-symmetric spaces.

The volume ratio of a convex body $C \subseteq \mathbf{R}^n$ is

$$\mathrm{vr}(C) = \inf_{\varepsilon \subseteq C} \left(\frac{\mathrm{vol}_n(C)}{\mathrm{vol}_n(\varepsilon)} \right)^{1/n}$$

where the infimum is taken over all ellipsoids ε that are contained in C. The volume ratio $\mathrm{vr}(E)$ of a space E is that of its unit ball.

Proposition 1.3. *Let C be a convex body in $\mathbf{R}^n$ such that the Euclidean sphere B_2^n is the ellipsoid of maximal volume contained in C. Then we have*

$$\mathrm{iq}(C) \leqq \mathrm{vr}(C) n \, \mathrm{vol}_n(B_2^n)^{1/n} \leqq c\sqrt{n}\, \mathrm{vr}(C)$$

where c is an absolute constant.

It was shown in [Schü$_1$, ST] that $\mathrm{vr}(C_p^n)$ for $1 \leqq p \leqq 2$ and $\mathrm{vr}(l_p^n \otimes_\pi l_p^n)$ for $1 \leqq p \leqq 2$ are uniformly bounded. Therefore Theorem 1.1 and Proposition 1.3 give that for these spaces $\mathrm{vol}_{n-1}(P_\xi(B_E))$ is essentially not dependent on the direction ξ. This can also be obtained by arguments of Ball and Bourgain [Ball]. Later we shall see that the isoperimetric quotient of the unit balls of l_p^n, $2 \leqq p < \infty$, are uniformly bounded although the volume ratios are not.

Lemma 1.4 [Had, pp. 161–163]. *Let C be a compact subset of $\mathbf{R}^n$ and $x_1, \ldots, x_n$ orthonormal vectors. Then we have*

$$\mathrm{vol}_n(C) \leqq \left| \prod_{i=1}^{n} \mathrm{vol}_{n-1}(P_{x_i}(C)) \right|^{1/(n-1)} .$$

Proof of Theorem 1.1. Let μ be the surface measure on ∂B_E and $N(x)$ the normal at x.

$$\operatorname{vol}_{n-1}(P_\xi(B_E)) = \frac{1}{2}\int_{\partial B_E} |\langle \xi, N(y)\rangle|\, d\mu(y)$$

$$= \frac{1}{2}\int_{\partial B_E} \operatorname*{Ave}_{\varepsilon\in\Gamma} |\langle (\varepsilon_i \xi_i)_{i=1}^n, N(y)\rangle|\, d\mu(y).$$

By the hypothesis we get

$$\operatorname{vol}_{n-1}(P_\xi(B_E)) \geqq \frac{1}{2c_1}\int_{\partial B_E} \left(\sum_{i=1}^n |\xi_i N(y)(i)|^2\right)^{1/2} d\mu(y)$$

$$\geqq \frac{1}{2c_1}\left(\sum_{i=1}^n |\xi_i|^2 \left|\int_{\partial B_E} |N(y)(i)|\, d\mu(y)\right|^2\right)^{1/2}$$

$$= \frac{1}{c_1}\operatorname{vol}_{n-1}(P_{e_1}(B_E)).$$

The last equality holds because we have for all $\pi \in \Pi$

$$\left\| \sum_{i=1}^n a_i e_i \right\| = \left\| \sum_{i=1}^n a_{\pi(i)} e_i \right\|$$

and consequently $\operatorname{vol}_{n-1}(P_{e_i}(B_E)) = \operatorname{vol}_{n-1}(P_{e_1}(B_E))$ for all $i = 1, \ldots, n$.

Thus, for all directions ξ we have

$$\frac{1}{c_1}\operatorname{vol}_{n-1}(P_{e_1}(B_E)) \leqq \operatorname{vol}_{n-1}(P_\xi(B_E)). \tag{1.5}$$

In a similar way we obtain that we have for all $\pi \in \Pi$

$$\operatorname{vol}_{n-1}(P_\xi(B_E)) \leqq \frac{c_2}{2}\int_{\partial B_E} \left(\sum_{i=1}^n |\xi_{\pi(i)} N(y)(i)|^2\right)^{1/2} d\mu(y).$$

Therefore

$$\operatorname{vol}_{n-1}(P_\xi(B_E)) \leqq \frac{c_2}{2}\left(\operatorname*{Ave}_{\pi\in\Pi} \left|\int_{\partial B_E} \left(\sum_{i=1}^n |\xi_{\pi(i)} N(y)(i)|^2\right)^{1/2} d\mu(y)\right|^2\right)^{1/2}$$

$$\leqq \frac{c_2}{2}\int_{\partial B_E} \left(\operatorname*{Ave}_{\pi\in\Pi} \sum_{i=1}^n |\xi_{\pi(i)} N(y)(i)|^2\right)^{1/2} d\mu(y)$$

$$= \frac{c_2}{2}\frac{1}{\sqrt{n}}\int_{\partial B_E} \left(\sum_{i=1}^n |N(y)(i)|^2\right)^{1/2} d\mu(y)$$

$$= \frac{c_2}{2} \frac{1}{\sqrt{n}} \operatorname{vol}_{n-1}(\partial B_E).$$

Combining this inequality and (1.5) we get

$$\max_{\zeta,\eta} \frac{\operatorname{vol}_{n-1}(P_\zeta(B_E))}{\operatorname{vol}_{n-1}(P_\eta(B_E))} \leqq \frac{c_1 c_2}{2} \frac{1}{\sqrt{n}} \frac{\operatorname{vol}_{n-1}(\partial B_E)}{\operatorname{vol}_{n-1}(P_{e_1}(B_E))}. \tag{1.6}$$

Now we apply Lemma 1.4. □

Proof of Proposition 1.3. Let $x \in \partial C$ and $N(x)$ the normal of ∂C at x. Since $B_2^n \subseteq C$ we have

$$\langle x, N(x) \rangle \geqq 1.$$

Therefore we get

$$\operatorname{vol}_n(C) = \frac{1}{n} \int_{\partial C} \langle x, N(x) \rangle d\mu \geqq \frac{1}{n} \operatorname{vol}_{n-1}(\partial C);$$

this implies

$$n \operatorname{vol}_n(C)^{1/n} \geqq \frac{\operatorname{vol}_{n-1}(\partial C)}{\operatorname{vol}_n(C)^{(n-1)/n}} = \operatorname{iq}(C),$$

$$n \operatorname{vol}_n(B_2^n)^{1/n} \operatorname{vr}(C) \geqq \operatorname{iq}(C).$$ □

Let $x \in \mathbf{R}^n$ and x^* the decreasing rearrangement of $|x_1|, |x_2|, |x_3|, \ldots, |x_n|$. We denote by

$$\max_k(x) = x_k^*.$$

Lemma 1.5. *Let $\{e_i\}_{i=1}^n$ be a 1-symmetric basis of E. Let*

$$a_i = \operatorname{vol}_{n-1}(\partial B_E)^{-1} \int_{\partial B_E} \max_i(N(y)) d\mu(y)$$

$$b_i = \left(\operatorname{vol}_{n-1}(\partial B_E)^{-1} \int_{\partial B_E} |\max_i(N(y))|^2 d\mu(y)\right)^{1/2} \qquad i = 1, \ldots, n$$

and let M_1 and M_2 be Orlicz functions that satisfy

$$\tfrac{1}{3}M_1^{*-1}\left(\frac{l}{n}\right) \leqq \left(\sum_{i=1}^{n} a_i\right)^{-1}\left\{\sum_{i=1}^{l} a_i + \left(l \sum_{i=l+1}^{n} |a_i|^2\right)^{1/2}\right\} \leqq 2M_1^{*-1}\left(\frac{l+1}{n}\right),$$

$$l = 1, \dots, n.$$

$$\tfrac{1}{3}M_2^{*-1}\left(\frac{l}{n}\right) \leqq \left(\sum_{i=1}^{n} b_i\right)^{-1}\left\{\sum_{i=1}^{l} b_i + \left(l \sum_{i=l+1}^{n} |b_i|^2\right)^{1/2}\right\} \leqq 2M_2^{*-1}\left(\frac{l+1}{n}\right),$$

Then we have for all directions ξ, $\| \xi \|_2 = 1$,

$$c_1 \frac{1}{n} \sum_{i=1}^{n} a_i \operatorname{vol}_{n-1}(\partial B_E) \| \xi \|_{M_1} \leqq \operatorname{vol}_{n-1}(P_\xi(B_E)) \leqq c_2 \frac{1}{n} \sum_{i=1}^{n} b_i \operatorname{vol}_{n-1}(\partial B_E) \| \xi \|_{M_2}$$

where c_1 *and* c_2 *are absolute constants.*

Lemma 1.6 [KS₁, KS₂]. *Let* $a_1 \geqq a_2 \geqq \cdots \geqq a_n > 0$ *with* $\Sigma_{i=1}^n a_i = n$. *Then there is an Orlicz function* M *with*

$$\tfrac{1}{3}M^{*-1}\left(\frac{l}{n}\right) \leqq \frac{1}{n}\left\{\sum_{i=1}^{l} a_i + \left(l \sum_{i=l+1}^{n} |a_i|^2\right)^{1/2}\right\} \leqq 2M^{*-1}\left(\frac{l+1}{n}\right) \quad \text{for } 1 \leqq l \leqq n$$

such that

$$c_1 \| x \|_M \leqq \operatorname*{Ave}_\pi \left(\sum_{i=1}^{n} |x_{\pi(i)} a_i|^2\right)^{1/2} \leqq c_2 \| x \|_M$$

where c_1 *and* c_2 *are absolute constants.*

Proof of Lemma 1.5. In the proof of Theorem 1.1 we have shown

$$\begin{aligned} \operatorname{vol}_{n-1}(P_\xi(B_E)) &\leqq \frac{1}{2}\int_{\partial B_E}\left(\sum_{i=1}^{n} |\xi_i N(y)(i)|^2\right)^{1/2} d\mu(y) \\ &\leqq \sqrt{2}\operatorname{vol}_{n-1}(P_\xi(B_E)). \end{aligned} \tag{1.4}$$

Because we have a symmetric basis we get

$$\begin{aligned} 2\sqrt{2}\operatorname{vol}_{n-1}(P_\xi(B_E)) &\geqq \operatorname*{Ave}_\pi \int_{\partial B_E}\left(\sum_{i=1}^{n} |\xi_{\pi(i)} N(y)((i))|^2\right)^{1/2} d\mu(y) \\ &= \operatorname*{Ave}_\pi \int_{\partial B_E}\left(\sum_{i=1}^{n} |\xi_{\pi(i)} \max_i(N(y))|^2\right)^{1/2} d\mu(y) \\ &\geqq \operatorname*{Ave}_\pi \left(\sum_{i=1}^{n} |\xi_{\pi(i)}|^2 \left|\int_{\partial B_E} \max_i(N(y)) d\mu(y)\right|^2\right)^{1/2} \end{aligned}$$

$$= \mathrm{vol}_{n-1}(\partial B_E) \operatorname*{Ave}_{\pi} \left(\sum_{i=1}^{n} |\xi_{\pi(i)} a_i|^2 \right)^{1/2}.$$

Now we can apply Lemma 1.6 and we obtain the left-hand inequality of Lemma 1.5. The right-hand inequality is obtained in the same way. By (1.4) and the symmetricity we have

$$\begin{aligned}
&\mathrm{vol}_{n-1}(P_\xi(B_E)) \\
&\leqq \frac{1}{2} \operatorname*{Ave}_{\pi} \int_{\partial B_E} \left(\sum_{i=1}^{n} |\xi_{\pi(i)} N(y)(i)|^2 \right)^{1/2} d\mu(y) \\
&= \frac{1}{2} \operatorname*{Ave}_{\pi} \int_{\partial B_E} \left(\sum_{i=1}^{n} |\xi_{\pi(i)} \max_i(N(y))|^2 \right)^{1/2} d\mu(y) \\
&\leqq \frac{1}{2} \mathrm{vol}_{n-1}(\partial B_E) \operatorname*{Ave}_{\pi} \left(\mathrm{vol}_{n-1}(\partial B_E)^{-1} \int_{\partial B_E} \sum_{i=1}^{n} |\xi_{\pi(i)} \max_i N(y)|^2 d\mu(y) \right)^{1/2} \\
&= \frac{1}{2} \mathrm{vol}_{n-1}(\partial B_E) \operatorname*{Ave}_{\pi} \left(\sum_{i=1}^{n} |\xi_{\pi(i)} b_i|^2 \right)^{1/2}.
\end{aligned}$$

Again, we apply Lemma 1.6. □

2. Projection bodies of Lorentz spaces

By Lemma 1.6, there is for each sequence $a_1 \geqq a_2 \geqq \cdots \geqq a_n > 0$ an Orlicz function M so that

$$\begin{aligned}
\tfrac{1}{5} M^{*-1}\left(\frac{l}{n}\right) &\leqq \left(\sum_{i=1}^{n} a_i \right)^{-1} \left\{ \sum_{i=1}^{l} a_i + \left(l \sum_{i=l+1}^{n} |a_i|^2 \right)^{1/2} \right\} \\
&\leqq 2M^{*-1}\left(\frac{l+1}{n}\right), \qquad l = 1, \ldots, n.
\end{aligned} \tag{2.1}$$

We say that such an Orlicz function is associated to the sequence a.

THEOREM 2.1. (i) *Let* $1 < q < \infty$ *and* $d > 0$. *Then there are constants* $c_1, c_2 > 0$ *so that we have for all sequences* $1 = a_1 \geqq a_2 \geqq \cdots \geqq a_n > 0$ *with* $\Sigma_{i=1}^{k} a_i \leqq d a_k$ *for all* $k = 1, \ldots, n$ *and all directions* ξ, $\|\xi\|_2 = 1$,

$$c_1 \cdot c(a) \frac{1}{n} \|\xi\|_M \leqq \frac{\mathrm{vol}_{n-1}(P_\xi(B^n_{a,q}))}{\mathrm{vol}_{n-1}(\partial B^n_{a,q})} \leqq c_2 \frac{1}{n} \frac{\|a\|_1}{\|a\|_2} \|\xi\|_M$$

where M is associated to a and

$$c(a) = \left(\sum_{l=1}^{n} a_l\right)^{1/q} \left(\sum_{l \leqq n/e} |a_l|^2 \left|\sum_{k \leqq n/\ln(n/l)} a_k\right|^{2(1-q)/q}\right)^{-1/2}.$$

(ii) *There are constants $c_1, c_2 > 0$ so that we have for all sequences $1 = a_1 \geqq a_2 \geqq \cdots \geqq a_n > 0$*

$$c_1 \frac{1}{n} \frac{\| a \|_1}{\| a \|_2} \| \xi \|_M \leqq \frac{\mathrm{vol}_{n-1}(P_\xi(B^n_{a,1}))}{\mathrm{vol}_{n-1}(\partial B^n_{a,1})} \leqq c_2 \frac{1}{n} \frac{\| a \|_1}{\| a \|_2} \| \xi \|_M$$

where M is associated to a.

If we put $a_i = i^{q/p-1}$ we get as an immediate consequence the following corollary.

Corollary 2.2. *Let $\xi \in \mathbf{R}^n$, $\| \xi \|_2 = 1$.*

(i) $2 < p < \infty$,

$$c_p^{-1} n^{-1/p'} \| \xi \|_{p'} \leqq \frac{\mathrm{vol}_{n-1}(P_\xi(B^n_{p,1}))}{\mathrm{vol}_{n-1}(\partial B^n_{p,1})} \leqq c_p n^{-1/p'} \| \xi \|_{p'}.$$

(ii) $1 < q < p/2 < \infty$,

$$c_{p,q}^{-1} |\ln(n)|^{(1-q)/p} n^{(q-p)/p} \| \xi \|_{p/(p-q)} \leqq \frac{\mathrm{vol}_{n-1}(P_\xi(B^n_{p,q}))}{\mathrm{vol}_{n-1}(\partial B^n_{p,q})}$$

$$\leqq c_{p,q} n^{(q-p)/p} \| \xi \|_{p/(p-q)}.$$

(iii) $\max(1, p/2) < q \leqq p < \infty$,

$$c_{p,q}^{-1} \frac{1}{\sqrt{n}} \leqq \frac{\mathrm{vol}_{n-1}(P_\xi(B^n_{p,q}))}{\mathrm{vol}_{n-1}(\partial B^n_{p,q})} \leqq c_{p,q} \frac{1}{\sqrt{n}}.$$

Corollary 2.2 shows in particular that the projection bodies of the unit balls of l^n_p are equivalent to the Euclidean spheres. We have better estimates for l^n_p.

Remark 2.3.

(i) $1 \leqq p \leqq n$,

$$\frac{1}{c} \sqrt{pn} \leqq \frac{\mathrm{vol}_{n-1}(\partial B^n_p)}{\mathrm{vol}_{n-1}(B^{n-1}_p)} \leqq c \sqrt{pn},$$

where c is a universal constant.

(ii) Let M be an Orlicz function that satisfies a Δ_2-condition. Then we have

$$\frac{1}{c}\sqrt{n} \leqq \frac{\mathrm{vol}_{n-1}(\partial B_M^n)}{\mathrm{vol}_{n-1}(B_M^{n-1})} \leqq c\sqrt{n}$$

where c depends only on the constant appearing in the Δ_2-condition.

To show the right-hand inequality of Remark 2.3(i) we estimate

$$\mathrm{vol}_{n-1}(\partial B_p^n) = 2\int_{B_p^{n-1}} x_n^{1-p}\left(\sum_{i=1}^n |x_i|^{2p-2}\right)^{1/2} dx$$

with $x_n = (1 - \Sigma_{i=1}^{n-1}|x_i|^p)^{1/p}$. The left-hand inequality follows from

$$\int_{B_p^{n-1}} |f(x)|^2 d\mu(x) \leqq \left(\int_{B_p^{n-1}} |f(x)|\, d\mu(x)\right)^{2/3}\left(\int_{B_p^{n-1}} |f(x)|^4 d\mu(x)\right)^{1/3}$$

with $f(x) = x_n^{1-p}(\Sigma_{i=1}^n |x_i|^{2p-2})^{1/2}$ and μ the normalized Lebesgue measure.

In order to prove part (ii) of Remark 2.3 we have to use inequalities of the type

$$M(\theta t) \leqq \theta^\gamma M(t).$$

This inequality is true provided M satisfies a Δ_2-condition. Besides this, the arguments are the same as for Theorem 2.1.

LEMMA 2.4 [HLP pp. 45–49]. *Let $x_1 \geqq x_2 \geqq \cdots \geqq x_n \geqq 0$ and $y_1 \geqq y_2 \geqq \cdots \geqq y_n \geqq 0$ such that*

$$\sum_{i=1}^n x_i = \sum_{i=1}^n y_i,$$
$$\sum_{i=1}^k x_i \geqq \sum_{i=1}^k y_i, \qquad k = 1, \ldots, n,$$

then there are numbers $d_r \geqq 0$ with $\Sigma_r d_r = 1$ and permutations π_r so that

$$y_k = \sum_r d_r x_{\pi_r(k)}, \qquad k = 1, \ldots, n.$$

LEMMA 2.5. *Let $\xi_1 \geqq \xi_2 \geqq \cdots \geqq \xi_n > 0$ and $\eta_1 \geqq \eta_2 \geqq \cdots \geqq \eta_n > 0$ with $\|\xi\|_2 = \|\eta\|_2 = 1$ and*

$$\eta_i/\eta_{i+1} \leqq \xi_i/\xi_{i+1}, \qquad i = 1, \ldots, n-1.$$

Then we have for all $z \in \mathbf{R}^n$

$$\operatorname*{Ave}_\pi\left(\sum_{i=1}^n |z_{\pi(i)}\eta_i|^2\right)^{1/2} \geqq \operatorname*{Ave}_\pi\left(\sum_{i=1}^n |z_{\pi(i)}\xi_i|^2\right)^{1/2}.$$

PROOF. We put $x_i = \xi_i^2$ and $y_i = \eta_i^2$ for $i = 1, \dots, n$. Now we verify that the hypothesis of Lemma 2.4 is satisfied. Since $\eta_i/\eta_{i+1} \leqq \xi_i/\xi_{i+1}$ there must be $k_0 \in \mathbf{N}$ so that

$$\eta_i \leqq \xi_i \qquad \text{for } i = 1, \dots, k_0,$$

$$\eta_i > \xi_i \qquad \text{for } i = k_0 + 1, \dots, n.$$

Clearly it follows that

$$\sum_{i=1}^{k} \eta_i^2 \leqq \sum_{i=1}^{k} \xi_i^2 \qquad \text{for } k = 1, \dots, k_0,$$

$$\sum_{i=k}^{n} \xi_i^2 \leqq \sum_{i=k}^{n} \eta_i^2 \qquad \text{for } k = k_0 + 1, \dots, n.$$

Since we assume that $\Sigma_{i=1}^n \xi_i^2 = \Sigma_{i=1}^n \eta_i^2$ we get also for $k = k_0 + 1, \dots, n$

$$\sum_{i=1}^{k} \eta_i^2 \leqq \sum_{i=1}^{k} \xi_i^2.$$

Therefore we can apply Lemma 2.4. There are numbers $d_r \geqq 0$ with $\Sigma_r d_r = 1$ and permutations σ_r so that

$$\eta_k^2 = y_k = \sum_r d_r x_{\sigma_r(k)} = \sum_r d_r \xi^2_{\sigma_r(k)}.$$

Therefore we get by using the inverse triangle inequality for $p = \frac{1}{2}$

$$\left\{\operatorname*{Ave}_{\pi}\left(\sum_{i=1}^{n} |z_{\pi(i)}\xi_i|^2\right)^{1/2}\right\}^2 = \sum_r d_r \left\{\operatorname*{Ave}_{\pi}\left(\sum_{i=1}^{n} |z_{\pi(i)}\xi_{\sigma_r(i)}|^2\right)^{1/2}\right\}^2$$

$$\leqq \left\{\operatorname*{Ave}_{\pi}\left(\sum_r d_r \sum_{i=1}^{n} |z_{\pi(i)}\xi_{\sigma_r(i)}|^2\right)^{1/2}\right\}^2$$

$$= \left\{\operatorname*{Ave}_{\pi}\left(\sum_{i=1}^{n} |z_{\pi(i)}\eta_i|^2\right)^{1/2}\right\}^2. \qquad \square$$

LEMMA 2.6. (i) *Let* $1 \leqq q < \infty$ *and* $1 = a_1 \geqq a_2 \geqq \cdots \geqq a_n > 0$ *such that* $\Sigma_{i=1}^k a_i \leqq cka_k$ *for all* $k = 1, \dots, n$. *Then we have for all* k, $1 \leqq k \leqq n/e$ *and all* $\alpha > 0$

$$\int_{\partial B^n_{a,q}} |\max_k(x)|^\alpha d\mu(x) \leqq c_{\alpha,q} \operatorname{vol}_{n-1}(\partial B^n_{a,q}) \left\| \sum_{i \leqq n/\ln(n/k)} e_i \right\|^{-\alpha}$$

where $c_{\alpha,q}$ *does not depend on the dimension* n.

(ii) *Let* $1 \leqq q < \infty$ *and* $1 = a_1 \geqq a_2 \geqq \cdots \geqq a_n > 0$ *such that* $\Sigma_{i=1}^k a_i \leqq cka_k$ *for all* $k = 1, \ldots, n$. *Then there is a constant* $d = d(\alpha, q, c)$ *so that*

$$\int_{\partial B^n_{a,q}} |\max_{[n/d]}(x)|^\alpha d\mu(x) \geqq \frac{1}{d} \operatorname{vol}_{n-1}(\partial B^n_{a,q}) \left\| \sum_{i=1}^n e_i \right\|^{-\alpha}.$$

One can actually prove that (ii) holds for a space with a symmetric basis. In order to show Lemma 2.6 we need the following lemma.

Lemma 2.7. *Let* $1 < q < \infty$ *and* $1 = a_1 \geqq a_2 \geqq \cdots \geqq a_n > 0$ *so that* $\Sigma_{i=1}^k a_i \leqq dka_k$, $k = 1, \ldots, n$. *Then we have for all* r, $1 \leqq r \leqq n$,

(i) $$\operatorname{vol}_{n-1}\left\{x \in \partial B^n_{a,q} \,\middle|\, \max_k(x) \geqq \frac{2}{\lambda(r)}\right\} \leqq \binom{n}{k} \left|1 - \frac{k}{dr}\right|^{(n-1)/q} \operatorname{vol}_{n-1}(\partial B^n_{a,q})$$

for $1 \leqq k \leqq r$,

(ii) $$\operatorname{vol}_{n-1}\left\{x \in \partial B^n_{a,q} \,\middle|\, \max_k(x) \geqq \frac{1}{\lambda(r)}\right\} = 0 \quad \textit{for } k > r.$$

Proof. (ii) is obvious. We use that $\lambda(r) < \lambda(r+1)$.

We show (i). We prove first that

$$\begin{aligned} &B^n_{a,q} \cap \{x \in \mathbf{R}^n \mid x_1, \ldots, x_k \geqq 2/\lambda(r)\} \\ &\subseteq \left|1 - \frac{k}{dr}\right|^{1/q} B^n_{a,q}\left(\frac{1}{\lambda(r)} \sum_{i=1}^k e_i\right) \cap \left\{x \in \mathbf{R}^n \,\middle|\, x_1, \ldots, x_k \geqq \frac{1}{\lambda(r)}\right\}. \end{aligned} \tag{2.2}$$

We have for x in the left-hand set of (2.2)

$$x_i - \frac{1}{\lambda(r)} \geqq \frac{1}{\lambda(r)}, \qquad i = 1, \ldots, k.$$

Since $\| x \|_{a,q} \leqq 1$ we also have that

$$\tilde{x} = \left(x_1 - \frac{1}{\lambda(r)}, \ldots, x_k - \frac{1}{\lambda(r)}, x_{k+1}, \ldots, x_n\right)$$

satisfies $\| \tilde{x} \|_{a,q} \leqq 1$. We claim that the first k coordinates of $\tilde{x}$ are among the r greatest coordinates of $\tilde{x}$, i.e. if $\tilde{x}^*$ denotes the decreasing rearrangement of $\tilde{x}$ there is a set $I = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, r\}$ so that

$$\tilde{x}^*_{i_j} = x_j - 1/\lambda(r), \qquad j = 1, \ldots, k. \tag{2.3}$$

Indeed, if one of those coordinates would not be among the r greatest then more than r coordinates of $\tilde{x}$ are greater than $1/\lambda(r)$ which contradicts the fact that $\| \tilde{x} \|_{a,q} \leqq 1$. Therefore we get

$$1 \geqq \sum_{i=1}^{n} a_i |x_i^*|^q \geqq \sum_{i \notin I} a_i |\tilde{x}_i^*|^q + \sum_{i \in I} a_i |\tilde{x}_i^* + 1/\lambda(r)|^q$$

$$\geqq \sum_{i \notin I} a_i |\tilde{x}_i^*|^q + \sum_{i \in I} a_i |\tilde{x}_i^*|^q + \sum_{i \in I} a_i |1/\lambda(r)|^q.$$

Thus we get

$$1 - \sum_{i \in I} a_i |1/\lambda(r)|^q \geqq \sum_{i=1}^{n} a_i |\tilde{x}_i^*|^q$$

or

$$\left\| x - \frac{1}{\lambda(r)} \sum_{i=1}^{r} e_i \right\|_{a,q} = \| \tilde{x} \|_{a,q} \leqq \left| 1 - \frac{k a_r}{\sum_{i=1}^{r} a_i} \right|^{1/q} \leqq \left| 1 - \frac{k}{dr} \right|^{1/q}.$$

Now we show that (2.2) implies Lemma 2.7(i). (2.2) implies that we have for all signs $\varepsilon_1, \ldots, \varepsilon_k$

$$B_{a,q}^n \cap \{x \in \mathbf{R}^n \mid \varepsilon_1 x_1, \ldots, \varepsilon_k x_k \geqq 2/\lambda(r)\}$$

$$\subseteq \left| 1 - \frac{k}{dr} \right|^{1/q} B_{a,q}^n \left(\frac{1}{\lambda(r)} \sum_{i=1}^{k} \varepsilon_i e_i \right) \cap \{x \in \mathbf{R}^n \mid \varepsilon_1 x_1, \ldots, \varepsilon_k x_k \geqq 1/\lambda(r)\}$$

or, after a shift by $(1/\lambda(r)) \sum_{i=1}^{k} \varepsilon_i e_i$,

$$B_{a,q}^n \left(-\frac{1}{\lambda(r)} \sum_{i=1}^{k} \varepsilon_i e_i \right) \cap \{x \in \mathbf{R}^n \mid \varepsilon_1 x_1, \ldots, \varepsilon_k x_k \geqq 1/\lambda(r)\}$$

$$\subseteq \left| 1 - \frac{k}{dr} \right|^{1/q} B_{a,q}^n \cap \{x \in \mathbf{R}^n \mid \varepsilon_1 x_1, \ldots, \varepsilon_k x_k \geqq 0\}$$

and therefore

$$B_{a,q}^n \left(-\frac{2}{\lambda(r)} \sum_{i=1}^{k} \varepsilon_i e_i \right) \cap \{x \in \mathbf{R}^n \mid \varepsilon_1 x_1, \ldots, \varepsilon_k x_k \geqq 0\}$$

(2.4)

$$\subseteq \left| 1 - \frac{k}{dr} \right|^{1/q} B_{a,q}^n \cap \{x \in \mathbf{R}^n \mid \varepsilon_1 x_1, \ldots, \varepsilon_k x_k \geqq 0\}.$$

Since $l_{a,q}^n$ has a 1-unconditional basis we have that

$$\bigcup_{\varepsilon}\left\{B^n_{a,q}\left(-\frac{2}{\lambda(r)}\sum_{i=1}^{k}\varepsilon_i e_i\right)\cap\{x\in\mathbf{R}^n\mid\varepsilon_1x_1,\ldots,\varepsilon_kx_k\geqq 0\}\right\}$$

is a convex set that is contained in $|1-k/dr|^{1/q}B^n_{a,q}$. Thus we get

$$\mathrm{vol}_{n-1}\left\{\partial B^n_{a,q}\cap\bigcup_{\varepsilon}\{x\in\mathbf{R}^n\mid\varepsilon_1x_1,\ldots,\varepsilon_kx_k\geqq 2/\lambda(r)\}\right\}$$

$$=\mathrm{vol}_{n-1}\partial\left\{\bigcup_{\varepsilon}\left\{B^n_{a,q}\left(-\frac{2}{\lambda(r)}\sum_{i=1}^{k}\varepsilon_ie_i\right)\cap\{x\in\mathbf{R}^n\mid\varepsilon_1x_1,\ldots,\varepsilon_kx_k\geqq 0\}\right\}\right\}$$

$$\leqq|1-k/dr|^{(n-1)/q}\,\mathrm{vol}_{n-1}(\partial B^n_{a,q}).$$

This implies

$$\mathrm{vol}_{n-1}\{x\in\partial B^n_{a,q}\mid\max_k(x)\geqq 2/\lambda(r)\}\leqq\binom{n}{k}|1-k/dr|^{(n-1)/q}\,\mathrm{vol}_{n-1}(\partial B^n_{a,q}).\ \square$$

Proof of Lemma 2.6. (i) We show the inequality for k with $1\leqq k\leqq n/e$. We fix k and r_k with $k\leqq r_k\leqq n$. We specify later what we choose for r_k. By $\binom{n}{k}\leqq(en/k)^k$ and Lemma 2.7 we get

$$\mathrm{vol}_{n-1}(\partial B^n_{a,q})^{-1}\int_{\partial B^n_{a,q}}|\max_k(x)|^\alpha d\mu(x)$$

$$=\mathrm{vol}_{n-1}(\partial B^n_{a,q})^{-1}\int_0^\infty\mathrm{vol}_{n-1}\{|\max_k(x)|^\alpha\geqq t\}dt$$

$$\leqq\lambda(r_k)^{-\alpha}+\sum_{r=k}^{r_k-1}2^\alpha\lambda(r)^{-\alpha}\binom{n}{k}\left|1-\frac{k}{d(r+1)}\right|^{(n-1)/q}$$

$$\leqq\lambda(r_k)^{-\alpha}+2^\alpha\left(\frac{en}{k}\right)^k\sum_{r=k}^{r_k-1}\lambda(r)^{-\alpha}\exp\left(-\frac{(n-1)k}{qd(r+1)}\right).$$

Because $\lambda(r_k)\leqq 2(r_k/r)\lambda(r)$ for $r\leqq r_k$ we get

$$\leqq 2^\alpha\lambda(r_k)^{-\alpha}\left\{1+\sum_{r=k}^{r_k-1}\exp\left(k\ln\left(\frac{en}{k}\right)+\alpha\ln\left(2\frac{r_k}{r}\right)-\frac{(n-1)k}{qd(r+1)}\right)\right\}.$$

Now we choose $r_k=c'n/\ln(n/k)$ and observe that, if $k\leqq n/e$ and $k\leqq c'n$,

$$\ln\left(\frac{r_k}{r}\right)=\ln\left(c'\frac{n}{r\ln(n/k)}\right)\leqq\ln\left(c'\frac{n}{k}\right)\leqq\ln\left(\frac{n}{k}\right),$$

$$\frac{(n-1)k}{qd(r+1)} \geqq \frac{1}{qd}\frac{(n-1)k\ln(n/k)}{c'n} \geqq \frac{1}{2qdc'}k\ln\left(\frac{n}{k}\right).$$

Therefore, if we choose c' sufficiently small, i.e.

$$\frac{1}{2qdc'} \geqq 4+\alpha\ln 2+\alpha,$$

we obtain

$$\begin{aligned}
&\mathrm{vol}_{n-1}(\partial B^n_{a,q})^{-1}\int_{\partial B^n_{a,q}}|\max_k|^\alpha d\mu(x)\\
&\quad\leqq 2^\alpha\lambda(r_k)^{-\alpha}\left\{1+\sum_{r=k}^{r_k-1}\exp\left(-2k\ln\left(\frac{n}{k}\right)\right)\right\}\\
&\leqq 2^\alpha\lambda(r_k)^{-\alpha}\left\{1+c'\frac{n}{\ln(n/k)}\left(\frac{n}{k}\right)^{-2k}\right\}\\
&\leqq c''\lambda(r_k)^{-\alpha}
\end{aligned}$$

where c'' depends on q, d, and α but not on n.

(ii) We show that

$$\mu\{x\in\partial B^n_{a,q}\mid \max_{[n/d]}(x)\geqq\lambda(n)^{-1}\}\geqq d^{-1} \tag{2.5}$$

from which (ii) follows immediately. We have

$$\begin{aligned}
\mathrm{vol}_{n-1}(\partial B^n_{a,q}) &= \int_{\partial B^n_{a,q}}\|x\|_{a,q}\,d\mu(x)\\
&\leqq\left(\int_{\partial B^n_{a,q}}\sum_{k=1}^n a_k|\max_k(x)|^q d\mu\right)^{1/q}\\
&\leqq\left(\sum_{k\leqq n/d}a_k\int_{\partial B^n_{a,q}}|\max_k(x)|^q d\mu\right)^{1/q}\\
&\quad+\left(\sum_{k>n/d}a_k\int_{\partial B^n_{a,q}}|\max_k(x)|^q d\mu\right)^{1/q}.
\end{aligned}$$

By (i) the first summand is less than

$$\begin{aligned} &c_q \operatorname{vol}_{n-1}(\partial B^n_{a,q}) \left(\sum_{k \leqq n/d} a_k \left(\sum_{i \leqq n/\ln(n/k)} a_i \right)^{-1} \right)^{1/q} \\ (2.6) \qquad &\leqq c_q \operatorname{vol}_{n-1}(\partial B^n_{a,q}) \left(\sum_{k \leqq n/d} a_k \frac{\ln(n/k)}{n} \frac{1}{a_{[n/\ln(n/k)]}} \right)^{1/q}. \end{aligned}$$

By the hypothesis $\Sigma_{i=1}^k a_i \leqq cka_k$, $k = 1, \ldots, n$, we get that $a_k \leqq \frac{1}{2} Da_{Dk}$, $k \leqq n/D$ if $D \geqq 2\exp(2c^2)$ (compare [Schü$_2$], Lemma 4.7). By this and by choosing d large enough the last expression can be bound by $\frac{1}{2}\operatorname{vol}_{n-1}(\partial B^n_{a,q})$. Please note that d only depends on c and q. We get

$$(2.7) \qquad \tfrac{1}{2}\operatorname{vol}_{n-1}(\partial B^n_{a,q}) \leqq \left(\sum_{k \geqq n/d} a_k \int_{\partial B^n_{a,q}} |\max_k(x)|^q d\mu \right)^{1/q}.$$

From this and Lemma 2.7(ii) the inequality (2.5) follows with a new constant d that depends only on c and q. □

Proof of Theorem 2.1. As in the proof of Theorem 1.1 we obtain

$$\begin{aligned} &\operatorname{vol}_{n-1}(P_\xi(B^n_{a,q})) \leqq \frac{1}{2} \int_{\partial B^n_{a,q}} \operatorname{Ave}_\pi \left(\sum_{i=1}^n |\xi_{\pi(i)} N(y)(i)|^2 \right)^{1/2} d\mu(y) \\ (2.8) \qquad &\leqq \sqrt{2}\operatorname{vol}_{n-1}(P_\xi(B^n_{a,q})). \end{aligned}$$

For all $y \in \mathbf{R}^n$, $y_1 > y_2 > \cdots > y_n > 0$ we have

$$(2.9) \qquad N(y) = \left(\frac{a_i |y_i|^{q-1}}{(\Sigma_{i=1}^n |a_i|y_i|^{q-1}|^2)^{1/2}} \right)_{i=1}^n.$$

(2.9) implies

$$\frac{N(y)(i)}{N(y)(i+1)} \geqq \frac{a_i}{a_{i+1}}.$$

We can apply Lemma 2.5 to $N(y)$ and $a/\|a\|_2$. We get

$$\operatorname{vol}_{n-1}(P_\xi(B^n_{a,q})) \leqq \tfrac{1}{2}\|a\|_2^{-1} \int_{\partial B^n_{a,q}} \operatorname{Ave}_\pi \left(\sum_{i=1}^n |\xi_{\pi(i)} a_i|^2 \right)^{1/2} d\mu(y).$$

We apply Lemma 1.6 and obtain the right-hand inequality of Theorem 2.1. Now we prove the left-hand inequality. We want to apply Lemma 1.5. It suffices to get estimates from below for

$$\mathrm{vol}_{n-1}(\partial B^n_{a,q})^{-1}\int_{\partial B^n_{a,q}} \max_i(N(y))d\mu(y)$$

$$= \mathrm{vol}_{n-1}(\partial B^n_{a,q})^{-1}\int_{\partial B^n_{a,q}} \frac{a_i(\max_i(y))^{q-1}}{\left(\sum_{k=1}^{n} |a_k(\max_k(y))^{q-1}|^2\right)^{1/2}} d\mu(y)$$

$$\geqq \frac{\left(\mathrm{vol}_{n-1}(\partial B^n_{a,q})^{-1}\int_{\partial B^n_{a,q}} |a_i(\max_i(y))^{q-1}|^{1/2} d\mu(y)\right)^2}{\mathrm{vol}_{n-1}(\partial B^n_{a,q})^{-1}\int_{\partial B^n_{a,q}} \left(\sum_{k=1}^{n} |a_k(\max_k(y))^{q-1}|^2\right)^{1/2} d\mu(y)}.$$

By Lemma 2.6 we get for $1 \leqq i \leqq n/d$

$$\geqq c_q \frac{a_i\lambda(n)^{1-q}}{\left(\sum_{k\leqq n/e} |a_k|^2 |\lambda([n/\ln(n/k)])|^{2-2q}\right)^{1/2}}.$$

Since $\lambda(k) = (\Sigma_{i=1}^k a_i)^{1/q}$ we get

$$\mathrm{vol}_{n-1}(\partial B^n_{a,q})^{-1}\int_{\partial B^n_{a,q}} \max_i(N(y))d\mu(y)$$

$$\geqq c'_q \frac{a_i\left(\sum_{k=1}^{n} a_k\right)^{(1-q)/q}}{\left(\sum_{k\leqq n/e} |a_k|^2\left(\sum_{l\leqq n/\ln(n/k)} a_l\right)^{(2-2q)/q}\right)^{1/2}}.$$

(ii) is easier to prove since $N(y) = a/\|a\|_2$. □

3. Successive quotients of quermassintegrals

Let C be a convex body. We denote by $W_i(C)$, $i = 0, \ldots, n$, the quermassintegrals [Had]. The Aleksandrov–Fenchel inequalities assure that the sequence of successive quotients of quermassintegrals is monotone. We show here that for certain convex bodies it is essentially constant (Corollary 3.4).

The following proposition is a combination of two known results.

Proposition 3.1. *Let E be the $\mathbf{R}^n$ equipped with a norm $\|\ \|_E$. Then we have*

$$n\frac{\mathrm{vol}_n(B_E)^{1/n}}{\mathrm{iq}(B_E)} = \frac{W_0(B_E)}{W_1(B_E)} \leqq \cdots \leqq \frac{W_{n-1}(B_E)}{W_n(B_E)} = \int_{\partial\Omega_n} \|x\|_{E^*} d\sigma(x).$$

LEMMA 3.2 (Aleksandrov–Fenchel) [Had, p. 202]. *For all convex bodies* $K \subset \mathbf{R}^n$ *we have*

$$W_i(K)^2 \geqq W_{i+1}(K) W_{i-1}(K), \qquad i = 1, \ldots, n-1.$$

LEMMA 3.3 [Had, p. 212]. *For all convex bodies* $K \subset \mathbf{R}^n$, $n > 1$, *we have*

$$W_{n-1}(K) = \tfrac{1}{2} \operatorname{vol}_n(\Omega_n) \int_{\partial \Omega_n} b(K, x) d\sigma(x)$$

where

$$b(K, x) = \inf\{t \mid \langle x, u\rangle \leqq t \text{ for all } u \in K\} - \sup\{t \mid \langle x, u\rangle \geqq t \text{ for all } u \in K\}.$$

PROOF OF PROPOSITION 3.1. Since $W_0(B_E) = \operatorname{vol}_n(B_E)$ and $W_1(B_E) = (1/n)\operatorname{vol}_{n-1}(\partial B_E)$ we get

$$\frac{W_0(B_E)}{W_1(B_E)} = n\,\frac{\operatorname{vol}_n(B_E)}{\operatorname{vol}_{n-1}(\partial B_E)} = n\,\frac{\operatorname{vol}_n(B_E)^{1/n}}{\operatorname{iq}(B_E)}.$$

The right-hand equality follows from Lemma 3.3 and the observation that $b(B_E, x) = 2 \| x \|_{E^*}$. The rest follows from Lemma 3.2. □

COROLLARY 3.4. *Let* $\max(1, p/2) < q \leqq p < \infty$. *Then we have*

$$\frac{1}{c_{p,q}} n^{1/2-1/p} \leqq \frac{W_0(B^n_{p,q})}{W_1(B^n_{p,q})} \leqq \cdots \leqq \frac{W_{n-1}(B^n_{p,q})}{W_n(B^n_{p,q})} \leqq c_{p,q} n^{1/2-1/p}.$$

PROOF. The left-hand inequality follows from Proposition 3.1, Corollary 2.2(iii) and Lemma 1.2 in [Schü$_1$]. Since we have $\| x \|_{p,q} \geqq \| x \|_p$ we get by dualization

$$\frac{W_{n-1}(B^n_{p,q})}{W_n(B^n_{p,q})} \leqq \int_{\partial\Omega_n} \| x \|_{p'} d\sigma(x) \leqq \left(\int_{\partial\Omega_n} \| x \|_{p'}^{p'} d\sigma(x) \right)^{1/p'}$$
$$= \left(n \int_{\partial\Omega_n} |x_1|^{p'} d\sigma(x) \right)^{1/p'} \leqq c_p n^{1/2-1/p}. \qquad \square$$

REMARK 3.5.

(i) [Had, p. 216] $W_i(B^n_\infty) = 2^{n-i} \operatorname{vol}_i(\Omega_i)$, $i = 1, \ldots, n$.

(ii) $W_0(B^n_1) = (1/\sqrt{n}) W_1(B^n_1)$.

(iii) $(1/c)\sqrt{\ln(n)/n} \leqq W_{n-1}(B^n_1)/W_n(B^n_1) \leqq c\sqrt{\ln(n)/n}$.

PROOF. (ii) We have

$$W_0(B_1^n) = \mathrm{vol}_n(B_1^n) = \frac{2^n}{n!} \quad \text{and} \quad W_1(B_1^n) = \frac{1}{n}\mathrm{vol}_{n-1}(\partial B_1^n) = \frac{2^n\sqrt{n}}{n!}.$$

(iii) By Proposition 3.1 we have

$$\frac{W_{n-1}(B_1^n)}{W_n(B_1^n)} = \int_{\partial\Omega_n} \| x \|_\infty \, d\sigma(x).$$

A lower bound for this integral was given in [FLM]. The same method gives also an upper bound. Also, Lemma 2.6 gives the upper bound. □

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